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Generation of clusters in complex dynamical networks via pinning control

Kezan Li[1,2], Michael Small[2] and Xinchu Fu[1,2]

[1] Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China
[2] Department of Electronic and Information Engineering, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

E-mail: xcfu@shu.edu.cn

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Abstract

Many real-world networks show community structure, i.e., groups (or clusters) of nodes that have a high density of links within them but with a lower density of links between them. In this paper, by applying feedback injections to a fraction of network nodes, various clusters are synchronized independently according to the community structure generated by the group partition of the network (*cluster synchronization*). This control is achieved by pinning (i.e. applying linear feedback control) to a subset of the network nodes. Those pinned nodes are selected not randomly but according to the topological structure of communities of a given network. Specifically, for a given group partition of a network, those nodes with direct connections between groups must be pinned in order to achieve cluster synchronization. Both the local stability and global stability of cluster synchronization are investigated. Taking the tree-shaped network and the most modular network as two particular examples, we illustrate in detail how the pinning strategy influences the generation of clusters. The simulations verify the efficiency of the pinning schemes used in this paper.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Networks appear everywhere and many complex dynamical processes in science, engineering and nature can be analyzed as evolution of networks. Some typical examples of dynamical networks include ecological and food webs [1], the world wide web (WWW) [2], transmission of diseases in population [3–6], neural networks [7] and social networks [8]. A complex network is often constituted by many elements with interactions between them. From the viewpoint of computer science, an individual element can be considered as a node, while the links between nodes represent the relationship between elements. Generally speaking, the dynamics of an individual node are determined by two factors: each nodes dynamical properties and the influence from its neighbors.

In 1960, Erdös and Rényi (ER) [9] proposed the random graph model to describe network structure, based on the assumption of completely random connections between nodes in a network. However, in the real world it is hard to find a network whose connections are completely random. In order to describe a transition from a regular model to a random model, in 1998 Watts and Strogatz (WS) [10] introduced the small-world network model, in which some connections are determined beforehand and the others are produced randomly according to certain probability. This model can show two common characteristics of many real-world networks: large clustering coefficient and short average path length. In 1999, Barabási and Albert [11] introduced another network generation model which generates a power-law distribution of node degrees. Since then, the so-called scale-free networks have been observed in many real-life complex networks. Roughly speaking, in a scale-free network, most nodes have very few connections yet only a few nodes have many connections.

During the past few decades, the synchronization of complex networks has been studied thoroughly and widely. Most existing works ([12–23]) have concentrated on complete synchronization, i.e., all states of nodes in a network are asymptotically equal in the phase space. Many other patterns of synchronization have also been defined and studied, such as cluster synchronization [24, 25], phase and lag synchronization [26], bubbling synchronization [27] and generalized synchronization [28]. In this paper, we will address the cluster synchronization of complex networks with community structure via pinning control.

Design of a suitable control strategy to steer a dynamical network toward an expected special state is a very interesting and significant topic in complex network control. In [29, 30], feedback pinning has been investigated for the control of spatiotemporal chaos in regular dynamical networks. For heterogeneous networks, the targeted pinning scheme, which aims to pin the most highly connected nodes, is more effective than random pinning applied to the same number of nodes [31, 32]. In [33], Lu has introduced an adaptive dynamical network by integrating the complex network model and adaptive technique, and discussed the synchronization and pinning control of this network. In [34], the authors have proven that a single controller could pin a coupled complex network to a homogenous solution, and given sufficient conditions to guarantee the convergence of the pinning process both locally and globally. Other research works about pinning control of complex networks can be seen in [35, 36] and many references cited therein. All these works consider how to pin a complex network to a homogeneous state, i.e., all nodes converge to a single state.

In this paper, we will discuss a more general problem: how to construct a suitable pinning control strategy for a complex network with community structure going to an inhomogeneous state (cluster synchronization), i.e., the nodes in the same group (community) converge to a single state (desired state) and different groups converge to different states. That is, the aim is that the nodes within each state are fully synchronized (to some dynamic state) while nodes drawn from separate clusters behave independently. When these desired states are non-chaotic, this problem comes down to the investigation of the control of chaos, which has become an active field in the study of nonlinear dynamics due to its potential applications [37]. On the other hand, when these desired states are chaotic, this study may find applications in communication engineering and other fields of science and technology [38].

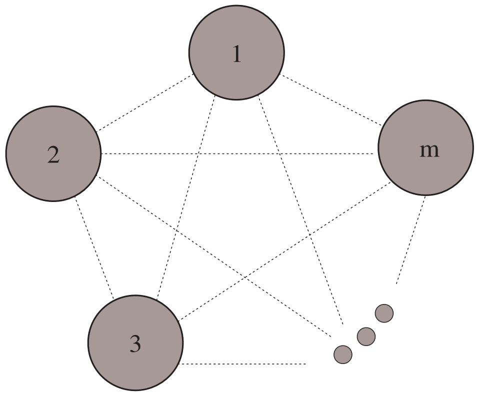

Figure 1. The community structure of a complex network. The *i*th community (group) in this network denoted by *i*th numbered gray circle. The dotted lines show the probable connections between the *m* communities.

In order to achieve cluster synchronization, we should pin at least those nodes with direct connections between groups in a network with community structure. With two kinds of control designed according to the community structure of a network (the controls are detailed below in (10) and (11)), we investigate both the local stability and global stability of cluster synchronization in section 3. Taking a tree-shaped network as a particular example in section 4, two clusters and three clusters are generated respectively by applying the appropriate pinning strategies. Another example of the most modular network, with large degree of community structure, is also presented to realize global synchronization of five clusters. The simulation results verify the efficiency of the pinning schemes used in this paper. Finally, we conclude in section 5.

2. Preliminaries

Most social networks show community structure [8, 39], i.e., groups of nodes that have a high density of links within them, yet with a lower density of links between groups. For social and biological networks at least, community structure is a common network property [40]. Supposing that a given network has been divided into m communities according to one criterion or another, then we can show the schematic diagram for it in figure 1.

Obviously, no community should be isolated according to the definition of community structure. At the same time, not every two communities in a network are required to have connections between them. Our following discussions will be based on these basic facts.

For a given network, there are different group partitions leading to different community structures. For example, a phone call network [41] can show different community structures according to different group partitions.

For simplicity, we now make some assumptions for a network with community structure as follows. Suppose that the number of nodes in this network is N, and $U_i = \{l_{i-1}+1, \ldots, l_i\}$, $i \in \{1, \ldots, m\}$ denotes the index set of all nodes in the ith community, where $l_0 = 0, l_m = N$ and $l_{i-1} < l_i, l_i \in \mathbb{N}$. Let $\widetilde{U}_i$ be the index set of all nodes in the ith community having direct connections to other communities. For each $j \in \widetilde{U}_i$, define V_{ij} as the set of nodes outside the ith community with direct connections to node j. And let $\sharp(U_i)$ denote the element number of the finite set U_i. According to these definitions, it is easy to see that $\widetilde{U}_i \subseteq U_i$ and $\bigcup_{i=1}^m U_i = \{1, \ldots, N\}$. Because of the property of community structure, in general $\bigcup_{i=1}^m \widetilde{U}_i$ is just a fraction of all nodes in a network. In view of this fact, we want to control these nodes in $\bigcup_{i=1}^m \widetilde{U}_i$ to generate cluster synchronization, i.e., the nodes in the same group U_i converge to a single state and different groups $U_i, U_j (i \neq j)$ converge to different states.

In this paper, we consider the dynamical network which consists of *N* identical linearly and diffusively coupled nodes, with each node being an *n*-dimensional dynamical system. The state equations of this network have the following form:

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma} x_j(t), \qquad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T \in \mathbb{R}^n$ are the state variables of node *i*; the function $f(\cdot)$ describes the local dynamics of nodes and is continuously differentiable and capable of showing abundant dynamical behaviors, including equilibrium points, periodic orbits and chaotic states. The constant $c > 0$ denotes the coupling strength and $\mathbf{\Gamma} \in \mathbb{R}^{n\times n}$ represents the inner-coupling matrix which is a constant $0 - 1$ matrix linking coupled variables. For simplicity, we assume that $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a diagonal matrix with $\gamma_i \geqslant 0$. The coupling matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ with zero-sum rows shows the coupling configuration of the network. If nodes *i* and *j* are connected, then $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$. The diagonal elements of the coupling matrix **A** are

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -k_i, \qquad i = 1, 2, \ldots, N, \tag{2}$$

where k_i denotes the degree of node *i*. In this paper, we suppose that the matrix **A** is irreducible, which means that the network is connected in the sense that there are no isolated clusters. In this case, we know that zero is an eigenvalue of **A** with multiplicity 1 and all the other eigenvalues are strictly negative.

3. Stability analysis for cluster synchronization under control

In this section, we will address the local stability and global stability of cluster synchronization of the considered network (1) under certain control. According to the control goal and assumptions presented in section 2, we first define the error variables as $e_i(t) = x_i(t) - s_j(t)$, for $j = 1, \ldots, m$ and $i = l_{j-1} + 1, \ldots, l_j$, where l_j are integers with $l_{j-1} < l_j$, $2 \leqslant m < N$, and $s_j(t)$ can be equilibrium point, periodic orbit or chaotic orbit in the phase space satisfying $\dot{s}_j(t) = f(s_j(t))$. Define $\mathcal{M} = (s_1(t), \ldots, s_1(t), \ldots, s_m(t), \ldots, s_m(t)) \subset \mathbb{R}^{N\times n}$ as the cluster synchronization manifold of network (1) under certain control, where $s_j(t)$ continuously appears $l_j - l_{j-1}$ times in the expression of $\mathcal{M}$.

The *N* nodes are said to achieve cluster synchronization, i.e., the manifold $\mathcal{M}$ is stable if

$$\lim_{t\to\infty} \|e_i(t)\| = 0, \qquad i = 1, 2, \ldots, N. \tag{3}$$

According to the above definition of error variables, we can first give the corresponding error system (without control) with respect to (1) as

$$\dot{e}_i(t) = f(x_i) - f(s_\tau) + c\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma} e_j + c\sum_{k=0, k\neq\tau-1}^{m-1}\sum_{j=l_k+1}^{l_{k+1}} a_{ij}\mathbf{\Gamma}(s_{k+1} - s_\tau),$$
$$\tau = 1, 2, \ldots, m, \qquad i = l_{\tau-1} + 1, \ldots, l_\tau. \tag{4}$$

Now let

$$\varphi_i(s_1, s_2, \ldots, s_m) = c \sum_{k=1}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_1), \tag{5}$$

for $i = 1, \ldots, l_1$, then we have the following result to define other φ_i, $i = l_{k-1}+1, \ldots, l_k$, $k = 1, 2, \ldots, m$, for an uniform expression of (4).

Lemma. *By using the uniform form in expression (5) of φ_i, then the error system (4) can be rewritten as*

$$\dot{e}_i(t) = f(x_i) - f(s_k) + c \sum_{j=1}^{N} a_{ij} \mathbf{\Gamma} e_j + \varphi_i(s_1, s_2, \ldots, s_m),$$
$$k = 1, 2, \ldots, m, \qquad i = l_{k-1} + 1, \ldots, l_k. \tag{6}$$

Proof. Here, we only show that

$$c \sum_{k=0, k\neq 1}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_2) = \varphi_i(s_1, \ldots, s_m),$$

for $i = l_1 + 1, \ldots, l_2$ (as $\tau = 2$ in the (4)), since other equalities ($\tau = 3, \ldots, m$) can be similarly verified. From the condition (2) of diffusive coupling, we have

$$\begin{aligned}
\sum_{k=0, k\neq 1}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_2) &= \sum_{j=1}^{l_1} a_{ij} \mathbf{\Gamma}(s_1 - s_2) + \sum_{k=2}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_2) \\
&= \sum_{j=1}^{l_1} a_{ij} \mathbf{\Gamma}(s_1 - s_2) + \sum_{k=2}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_1) - \sum_{k=2}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_2 - s_1) \\
&= \sum_{j=l_1+1}^{l_2} a_{ij} \mathbf{\Gamma}(s_2 - s_1) + \sum_{k=2}^{m-1} \sum_{j=l_k+1}^{l_{k+1}} a_{ij} \mathbf{\Gamma}(s_{k+1} - s_1) \\
&= \varphi_i(s_1, \ldots, s_m)/c,
\end{aligned}$$

as required. □

Obviously, according to the assumptions in section 2, we have

$$\varphi_i(s_1, s_2, \ldots, s_m) = 0, \qquad i \in \left(\bigcup_{j=1}^{m} \widetilde{U}_j \right)^c, \tag{7}$$

where $\left(\bigcup_{j=1}^{m} \widetilde{U}_j\right)^c = \{1, \ldots, N\} \setminus \bigcup_{j=1}^{m} \widetilde{U}_j$.

Now, let $u_i(t) \in \mathbb{R}^n$, $i = 1, \ldots, N$ be the control inputs; then the controlled dynamical network (1) can be described by

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} a_{ij} \mathbf{\Gamma} x_j(t) + u_i(t), \qquad i = 1, 2, \ldots, N. \tag{8}$$

From (6), we know that its corresponding error system can be described as

$$\dot{e}_i(t) = f(x_i) - f(s_k) + c \sum_{j=1}^{N} a_{ij} \mathbf{\Gamma} e_j + \varphi_i(s_1, s_2, \ldots, s_m) + u_i(t),$$
$$k = 1, 2, \ldots, m, \qquad i = l_{k-1} + 1, \ldots, l_k. \tag{9}$$

In view of the special property of community equipped to network (1), we design two kinds of controller to realize cluster synchronization. The first kind is of local feedback controller which is designed as

$$\begin{aligned} u_i(t) &= -cd_i\mathbf{\Gamma}e_i - \varphi_i, \qquad i \in \bigcup_{j=1}^{m} \widetilde{U}_j, \\ u_i(t) &= 0, \qquad i \in \left(\bigcup_{j=1}^{m} \widetilde{U}_j\right)^c, \end{aligned} \tag{10}$$

where $d_i > 0$ is the feedback control gain.

The other one is of nonlocal feedback controller constructed as

$$\begin{aligned} u_i(t) &= -cd_i\mathbf{\Gamma}e_i - c\sum_{j\in V_{ki}} a_{ij}\mathbf{\Gamma}e_j - \varphi_i, \qquad i \in \widetilde{U}_k, \\ u_i(t) &= 0, \qquad i \in U_k\backslash\widetilde{U}_k, \end{aligned} \tag{11}$$

where $k \in \{1, 2, \ldots, m\}$. It is easy to verify that the manifold $\mathcal{M}$ is an invariant manifold of the network (1) under the control given by (10) or (11). One can also intuitively understand the function of the controllers u_i in (10) and (11) in the following way. The terms φ_i are devoted to weakening the mutual influences among clusters at the intersection nodes, while the remainder terms (feedback control) are designed to synchronize all nodes in the same cluster.

Firstly, we consider the global stability of the cluster manifold $\mathcal{M}$ of the dynamical network (8) under control given by (10). Let $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$, with $d_i = 0$ for all $i \in \left(\bigcup_{j=1}^{m} \widetilde{U}_j\right)^c$, and $\mathbf{B} = \mathbf{A} - \mathbf{D} = (b_{ij})_{N\times N}$. Since $\mathbf{B}$ is a diagonally dominant matrix, we know that its eigenvalues satisfy $0 > \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$.

Theorem 1. *For given* c *and* $\{d_i\}_{i=1}^N$*, if there exist two positive definite matrices* $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_n)$, $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)$ *and a constant* $\xi > 0$ *such that*

$$(x - y)^T\mathbf{P}\big(f(x) - f(y) - \Delta\mathbf{\Gamma}(x - y)\big) \leqslant -\xi(x - y)^T(x - y), \tag{12}$$

for every $x, y \in \mathbb{R}^n$*, and* $c\lambda_1 + \delta_k < 0$ *for all* $k \in \{1, 2, \ldots, n\}$*, then the cluster manifold* $\mathcal{M}$ *of the dynamical network (8) under control (10) is globally stable.*

Proof. Define a Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^T\mathbf{P}e_i(t). \tag{13}$$

Then, its derivative along the solutions of the error system (9) under the control (10) is

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{l_1} e_i^T\mathbf{P}\left(f(x_i) - f(s_1) + c\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}e_j\right) + \cdots \\ &\quad + \sum_{i=l_{m-1}}^{N} e_i^T\mathbf{P}\left(f(x_i) - f(s_m) + c\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}e_j\right) \\ &= \sum_{i=1}^{l_1} e_i^T\mathbf{P}(f(x_i) - f(s_1) - \Delta\mathbf{\Gamma}e_i) + \cdots + \\ &\sum_{i=l_{m-1}}^{N} e_i^T\mathbf{P}(f(x_i) - f(s_m) - \Delta\mathbf{\Gamma}e_i) + \sum_{i=1}^{N} e_i^T\mathbf{P}\left(c\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma}e_j + \Delta\mathbf{\Gamma}e_i\right). \end{aligned} \tag{14}$$

Now denote $\widetilde{e}^k = (e_{k1}, \ldots, e_{kN})^T, k = 1, 2, \ldots, n$; then we have

$$\sum_{i=1}^{N} e_i^T \mathbf{P}(\Delta\mathbf{\Gamma} e_i) = \sum_{i=1}^{N}\sum_{k=1}^{n} e_{ki}^2 p_k \delta_k \gamma_k = \sum_{k=1}^{n} p_k\gamma_k \sum_{i=1}^{N} e_{ki}^2 \delta_k = \sum_{k=1}^{n} p_k\gamma_k (\widetilde{e}^k)^T (\delta_k \mathbf{I_N})\widetilde{e}^k, \tag{15}$$

where $\mathbf{I_N}$ is the N-order identity matrix and

$$\begin{aligned}\sum_{i=1}^{N} e_i^T \mathbf{P}\left(c\sum_{j=1}^{N} b_{ij}\mathbf{\Gamma} e_j\right) &= \sum_{i=1}^{N}\sum_{j=1}^{N} c b_{ij} e_i^T \mathbf{P\Gamma} e_j = \sum_{i=1}^{N}\sum_{j=1}^{N} c b_{ij} \sum_{k=1}^{n} e_{ki} p_k \gamma_k e_{kj} \\ &= \sum_{k=1}^{n} p_k\gamma_k \sum_{i=1}^{N}\sum_{j=1}^{N} c b_{ij} e_{ki} e_{kj} \\ &= \sum_{k=1}^{n} p_k\gamma_k (\widetilde{e}^k)^T (c\mathbf{B})\widetilde{e}^k. \end{aligned}\tag{16}$$

Substituting (15) and (16) into (14) and noting the assumptions gives that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant -\xi \sum_{i=1}^{N} e_i^T e_i \leqslant -\frac{2\xi}{\max\limits_i\{p_i\}} V(t).$$

It follows that

$$V(t) = O(\mathrm{e}^{-2\xi t/\max\limits_i\{p_i\}}).$$

Therefore, we get the global stability of the cluster manifold $\mathcal{M}$. □

From (12) in theorem 1, we can see that if $\delta_k, k = 1, 2, \ldots, n$ are large enough, then this condition can always be satisfied. But on the other hand, too many large δ_k will destroy the second condition $c\lambda_1 + \delta_k < 0$. So, a balance between them is needed for guaranteeing the global stability of the cluster manifold. This further indicates that the global stability is governed synchronously by both the network properties and dynamical behavior of individual nodes.

Next, we will address the local and global stability of the cluster manifold $\mathcal{M}$ of dynamical network (8) under control given by (11). Differentiating along the manifold $\mathcal{M}$, the controlled error system (9) can be further described by

$$\begin{aligned} \dot{e}_i &= Df(s_k)e_i + c\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma} e_j, \qquad i \in U_k\backslash\widetilde{U}_k, \\ \dot{e}_i &= Df(s_k)e_i + c\sum_{j=1}^{N} a_{ij}\mathbf{\Gamma} e_j - c d_i \mathbf{\Gamma} e_i - c\sum_{j\in V_{ki}} a_{ij}\mathbf{\Gamma} e_j, \qquad i \in \widetilde{U}_k, \end{aligned}\tag{17}$$

where $k \in \{1, 2, \ldots, m\}$ and $Df(s_k) \in \mathbb{R}^{n\times n}$ are the Jacobian matrices of the function $f(\cdot)$ at s_k.

Now define $\mathbf{E}_i(t) = (0, \ldots, 0, e_{l_{i-1}+1}, \ldots, e_{l_i}, 0, \ldots, 0)^T$, and $\mathbf{E}(t) = \sum_{i=1}^{m} \mathbf{E}_i(t)$ where $\mathbf{E}, \mathbf{E}_i \in \mathbb{R}^{N\times n}$. It is obvious that $\mathbf{E} = \mathbf{0}$ if and only if $\mathbf{E}_i = \mathbf{0}$ for all $i \in \{1, 2, \ldots, m\}$.

Let $\widetilde{\mathbf{A}} = (\widetilde{a}_{ij})_{N\times N}$ with $\widetilde{a}_{ii} = 0, i = 1, \ldots, N$, and $\widetilde{a}_{ij} = 1$ as $j \in V_{ki}$ with $i \in \widetilde{U}_k$; otherwise $\widetilde{a}_{ij} = 0$. Let $\widetilde{\mathbf{B}} = \mathbf{A} - \mathbf{D} - \widetilde{\mathbf{A}} = (\widetilde{b}_{ij})_{N\times N}$; then the controlled error system (17) can be rewritten as

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_1 + \dot{\mathbf{E}}_2 + \cdots + \dot{\mathbf{E}}_m = \sum_{i=1}^{m} \mathbf{E}_i Df(s_i)^T + c\widetilde{\mathbf{B}}\mathbf{E}\boldsymbol{\Gamma}$$
$$= \sum_{i=1}^{m} (\mathbf{E}_i Df(s_i)^T + c\widetilde{\mathbf{B}}\mathbf{E}_i\boldsymbol{\Gamma}). \tag{18}$$

Similarly, the matrix $\widetilde{\mathbf{B}}$ is negative definite because of its diagonal dominance. To investigate the stability of the solution $\mathbf{E} = \mathbf{0}$, it only needs to consider the stability of the solution $\mathbf{E}_i = \mathbf{0}$ of the following m systems:

$$\dot{\mathbf{E}}_i = \mathbf{E}_i Df(s_i)^T + c\widetilde{\mathbf{B}}\mathbf{E}_i\boldsymbol{\Gamma}, \qquad i = 1, 2, \ldots, m. \tag{19}$$

Let $0 > \widetilde{\lambda}_1 \geqslant \widetilde{\lambda}_2 \geqslant \cdots \geqslant \widetilde{\lambda}_N$ be the eigenvalues of matrix $\widetilde{\mathbf{B}}$ and assume that

$$\widetilde{\mathbf{B}}\phi_k = \widetilde{\lambda}_k\phi_k, \qquad k = 1, 2, \ldots, N. \tag{20}$$

Then perform an invertible transformation to $\mathbf{E}_i$ by

$$\mathbf{E}_i = \Phi\mathbf{Y}_i, \qquad i = 1, 2, \ldots, m, \tag{21}$$

where $\Phi = [\phi_1, \phi_2, \ldots, \phi_N] \in \mathbb{R}^{N\times N}$. Under this transformation, we get

$$\dot{\mathbf{Y}}_i = \mathbf{Y}_i Df(s_i)^T + c\Lambda\mathbf{Y}_i\boldsymbol{\Gamma}, \qquad i = 1, 2, \ldots, m, \tag{22}$$

where $\Lambda = \mathrm{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N)$. To study the stability of the m systems (22), it only needs to consider the stability of the following m systems:

$$\dot{\mathbf{Z}} = \mathbf{Z}Df(s_i)^T + c\Lambda\mathbf{Z}\boldsymbol{\Gamma}, \qquad i = 1, 2, \ldots, m, \tag{23}$$

where $\mathbf{Z} = (z_1, z_2, \ldots, z_N)^T \in \mathbb{R}^{N\times n}$.

It is easy to see that (23) further gives

$$\dot{z}_j = (Df(s_i) + c\widetilde{\lambda}_j\boldsymbol{\Gamma})z_j, \tag{24}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, N$. Similarly, in order to study the stability for (24), we only need to consider the stability of the following $m \times N$ subsystems:

$$\dot{z} = (Df(s_i) + c\widetilde{\lambda}_j\boldsymbol{\Gamma})z, \tag{25}$$

where $z \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, N$. By the above analysis, the stabilities of the systems (17) and (25) are equivalent to each other.

Now for a given $i \in \{1, 2, \ldots, m\}$, let $\mu_{ij}(t)$, $j = 1, 2, \ldots, n$ denote all the eigenvalues of the matrix $(1/2)(Df(s_i(t)) + Df(s_i(t))^T)$, $\mu(t) = \max_{1\leqslant i\leqslant m}\max_{1\leqslant j\leqslant n}\{\mu_{ij}(t)\}$ and $\gamma = \max_{1\leqslant i\leqslant n}\{\gamma_i\}$. Then we have the following results.

Theorem 2. *For given c and $\{d_i\}_{i=1}^N$, if there exists a constant $\rho > 0$ such that $\mu(t) < -c\widetilde{\lambda}_1\gamma - \rho$ for all $t > 0$, then the cluster manifold $\mathcal{M}$ of the dynamical network (8) under control (11) is locally stable.*

Proof. Since

$$\frac{1}{2}\frac{\mathrm{d}z(t)^T z(t)}{\mathrm{d}t} = z(t)^T[Df(s_i(t)) + c\widetilde{\lambda}_j\boldsymbol{\Gamma}]z(t)$$
$$= z(t)^T[(1/2)(Df(s_i(t)) + Df(s_i(t))^T) + c\widetilde{\lambda}_j\boldsymbol{\Gamma}]z(t), \tag{26}$$

by using the assumptions in this theorem, then from (26) we obtain

$$\frac{1}{2}\frac{\mathrm{d}z(t)^T z(t)}{\mathrm{d}t} \leqslant z(t)^T[\mu(t) + c\widetilde{\lambda}_1\gamma]z(t) < -\rho z(t)^T z(t), \tag{27}$$

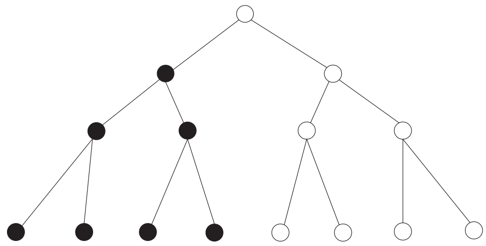

Figure 2. A group partition of a tree-shaped network leading to two communities. The black circles constitute one community U_1, while the white circles the other one U_2.

which implies that $z(t)^T z(t) = O(\mathrm{e}^{-2\rho t})$. This further means that the cluster synchronization manifold is stable locally. $\square$

Remark 1. Generally, the matrix $Df(s_i(t))$ is uniformly bounded, that is, there exists a constant $\beta \in \mathbb{R}$ such that $Df(s_i(t)) < \beta \mathbf{I_n}, \forall t > 0$. Therefore from (26), if there exists a constant $\rho > 0$ such that $\beta < -c\widetilde{\lambda}_1 \gamma - \rho$, then the cluster manifold is stable locally.

Remark 2. Substituting $\widetilde{\mathbf{B}}$ for $\mathbf{B}$ in (16), the global stability of cluster manifold of dynamical network (8) with control (11) can be similarly obtained with the conditions (12) and $c\widetilde{\lambda}_1 + \delta_k < 0$ for all $k \in \{1, 2, \ldots, n\}$.

4. Numerical simulations and analysis with the pinning control

4.1. The tree-shaped network

In this section, we take a tree-shaped network as a simple example to verify the efficiency of the pinning scheme introduced in previous sections. Although the topology structure of this network is simple, the exhibition of community structure is quite clear. So the tree-shaped networks could be considered as a typical class of networks holding community structure. For simplicity, here the considered network includes $N = 15$ nodes, whose topology structure is shown in figure 2. And the local dynamics of nodes are controlled by the Lorenz oscillator described by

$$\begin{cases} \dfrac{\mathrm{d}x_{1i}}{\mathrm{d}t} = a_1(x_{2i} - x_{1i}) \\ \dfrac{\mathrm{d}x_{2i}}{\mathrm{d}t} = a_2 x_{1i} - x_{2i} - x_{1i}x_{3i}, \qquad i = 1, 2, \ldots, 15, \\ \dfrac{\mathrm{d}x_{3i}}{\mathrm{d}t} = x_{1i}x_{2i} - a_3 x_{3i} \end{cases} \tag{28}$$

where $a_1 = 10, a_2 = 28, a_3 = (8/3)$. We choose matrix diag(1, 1, 1) as the inner-coupling matrix $\mathbf{\Gamma}$. And the control gains d_i to controlled nodes have the same values, i.e., $d_i = d$.

Next, we want to generate two kinds of cluster synchronization of this network via pinning control provided by (10). These two kinds of cluster synchronization are first based on two kinds of group partition on the network.

The first group partition ($m = 2$) is also shown in figure 2. Under this partition, we get two communities U_1 and U_2 with $\sharp(U_1) = 7$ and $\sharp(U_2) = 8$. According to the analysis in section 2 and 3, we should control two nodes in this network to achieve cluster synchronization, because

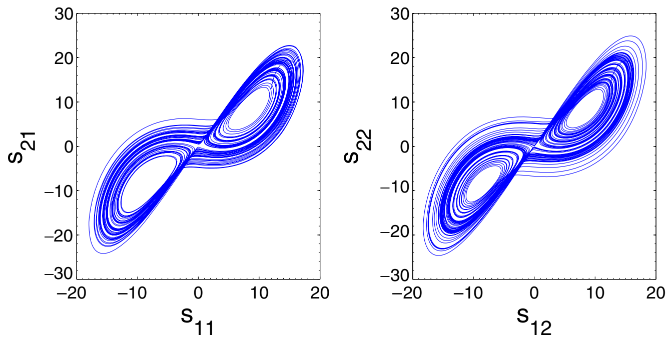

Figure 3. The desired two different states s_1 and s_2 both with chaotic behaviors in their two-dimension phase space.

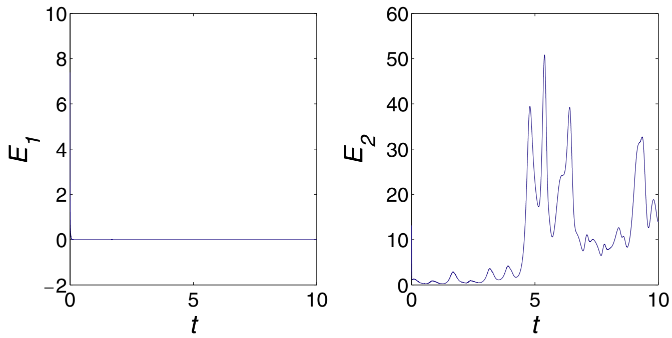

Figure 4. E_1 denotes the error of cluster synchronization for the controlled network and E_2 the error between the two communities U_1 and U_2, corresponding to $c = 750$ and $d = 20$.

$\sharp(\widetilde{U}_1 \bigcup \widetilde{U}_2) = 2$. Now define error E_1 of cluster synchronization for this controlled network and E_2 the error between the two communities U_1 and U_2 as

$$E_1(t) = \frac{1}{N}\left(\sum_{i\in U_1}\|x_i(t) - s_1(t)\| + \sum_{i\in U_2}\|x_i(t) - s_2(t)\|\right),$$
$$E_2(t) = \|x_{i_0}(t) - x_{j_0}(t)\|, \qquad i_0 \in U_1,\ j_0 \in U_2, \tag{29}$$

where s_1 and s_2 are chosen as two different chaotic orbits of the Lorenz system (see figure 3), with initial conditions (12, 13, 14) and (12.5, 13.5, 14.5), respectively. Certainly, we can also choose s_1 and s_2 as equilibrium points or periodic orbits. It is clear that the cluster synchronization is achieved if E_1 converges to zero and E_2 does not as $t \to \infty$. Under $c = 750$ and $d = 20$, figure 4 gives a realization of cluster synchronization via pinning control (10), whose global stability will be guaranteed by theorem 1. From [42], there exist constants $M_1 = 28.9180$ and $M_2 = 56.9180$ such that $\|s_{1i}\| \leqslant M_1$, $\|s_{2i}\| \leqslant M_1$ and $\|s_{3i}\| \leqslant M_2$. It then follows $e_i^T(f(x_i) - f(s_i)) \leqslant \left[-a_1 + \frac{a_1+a_2+M_2}{2} + \frac{M_1}{2}\right]e_i^T e_i \approx 51.918 e_i^T e_i$. So, the inequality (12) is satisfied with the setting $\mathbf{P} = \mathbf{I_3}$ and $\Delta = \delta\mathbf{I_3}$ with $\delta = 70.1$. In addition, the injection of two controllers with $d = 20$ leads to $\lambda_1 = -0.0938$. Therefore, we further obtain the second condition $c\lambda_1 + \delta = -0.25 < 0$ in theorem 1 for global stability of this two-cluster synchronization.

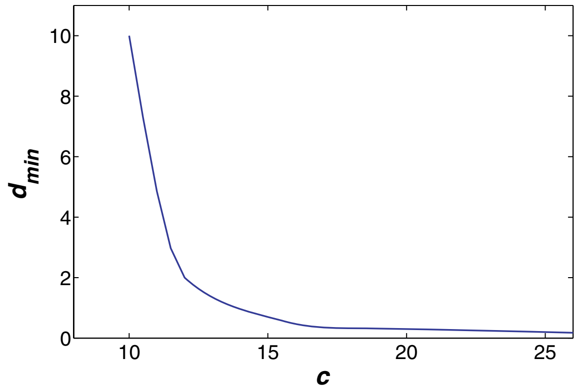

Figure 5. The minimal control gain $d_{\min} > 0$ for the achievement of two clusters synchronization with respect to coupling strength c.

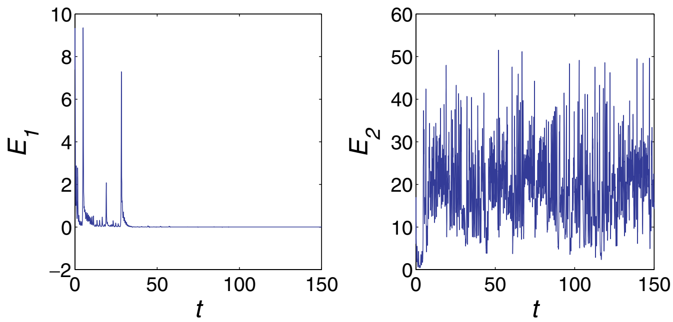

Figure 6. E_1 denotes the error of cluster synchronization for the controlled network and E_2 the error between the two communities U_1 and U_2, corresponding to $c = 12$ and $d = 3$.

From above, we see that very large coupling strength c should be required to meet the condition in theorem 1 to hold the global stability of cluster synchronization. In fact, the condition (12) is too strong (as mentioned also in [34]), while there is a quite loose one $(x - y)^T \mathbf{P}\big(f(x) - f(y)\big) \leqslant -\xi(x - y)^T(x - y)$, which is valid for many chaotic oscillators with many x, y. This implies that we can generally choose small δ_k to get the inequality (12). Then, small coupling strength c is just needed to satisfy the second condition $c\lambda_1 + \delta_k < 0$ for achieving the cluster synchronization globally. In view of this rough deduction, we now present a simulation-based analysis to show this case also arises in this example. By simulations, we find that there exists a constant $c^* = 9.98$ such that the network cannot be controlled successfully when the coupling strength c is smaller than c^*, even the control gain d is very large. On the other hand, for any given $c \geqslant c^*$, there exists the minimal control gain $d_{\min} > 0$ for the achievement of cluster synchronization. Moreover, very small value of d is needed for the achievement of control when c is relatively large.

Figure 5 gives the values $d_{\min}$ for the successful control with respect to different and relatively small coupling strength $c \geqslant c^*$. For example, with $c = 12$ and $d = 3$, figure 6 displays explicitly that the network is successfully controlled by the two feedback controllers

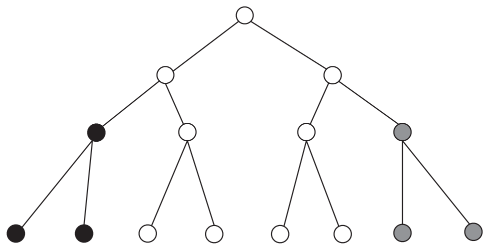

Figure 7. A group partition of a tree-shaped network leading to three communities. The black circles and gray circles respectively constitute two communities U_1 and U_2, while the white circles the third one U_3.

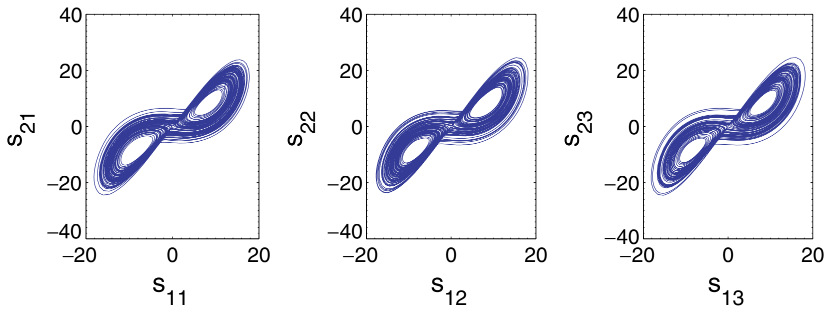

Figure 8. The desired three different states s_1, s_2 and s_3 all with chaotic behaviors in their two-dimension phase space.

provided by (10) in section 3. Now we consider $m = 3$ and a corresponding partition to the aforementioned tree-shaped network is shown by figure 7. Under this partition, we have $\sharp(U_1) = 3, \sharp(U_2) = 9$ and $\sharp(U_3) = 3$. And, we should control four nodes in this network to realize cluster synchronization because $\sharp(\widetilde{U}_1 \bigcup \widetilde{U}_2 \bigcup \widetilde{U}_3) = 4$. The definitions of error E_1 and E_2 for this controlled network are similar to (29), while E_3 and E_4 are defined by

$$E_3(t) = \|x_{i_0}(t) - x_{j_0}(t)\|, \qquad i_0 \in U_1, \quad j_0 \in U_3,$$
$$E_4(t) = \|x_{i_0'}(t) - x_{j_0'}(t)\|, \qquad i_0' \in U_2, \quad j_0' \in U_3.$$

Following the similar analysis process for dealing with the above two clusters, we can also choose large coupling strength c from theorem 1 to realize these three clusters globally. Here, we omit this analogous analysis just for simplicity, but also show the probability of successful control with relatively small c, based on a simulation analysis. In this simulation, we still choose s_1, s_2 and s_3 as different chaotic orbits (see figure 8) from the Lorenz system with initial conditions (12, 13, 14), (12.5, 13.5, 14.5) and (13, 14, 15), respectively. Obviously, cluster synchronization is achieved if E_1 converges to zero and E_2, E_3, E_4 do not. Figure 9 shows the values $d_{\min}$ for the control gain with different coupling strength $c \geqslant c^*$. For this controlled network, we get $c^* = 3.51$. From figure 10, we know that the network is successfully controlled by the four feedback controllers provided by (10) in section 3, even the coupling strength is small.

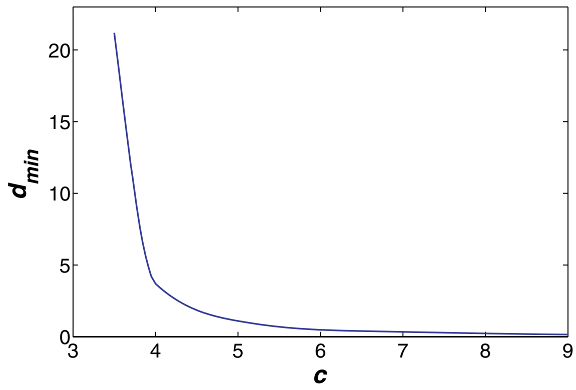

Figure 9. The minimal control gain $d_{\min} > 0$ for the achievement of three clusters synchronization with respect to coupling strength c.

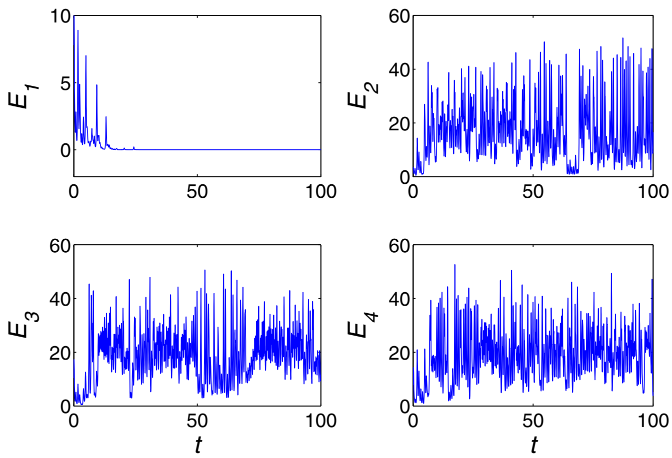

Figure 10. E_1 denotes the error of cluster synchronization for the controlled network, E_2 the error between U_1 and U_2, E_3 the error between U_1 and U_3 and E_4 the error between U_2 and U_3, corresponding to $c = 6$ and $d = 1$.

4.2. The most modular network

In order to consider a more definite and typical community network, we now address the most modular network [43], whose degree of community structure is quantified by the network modularity. The modularity of a partition of a network is defined as

$$Q = \sum_{i=1}^{m} \left[\frac{L_i}{L} - \left(\frac{D_i}{2L} \right)^2 \right], \tag{30}$$

where L denotes the total number of links in the network, L_i the number of links inside community i and D_i the total degree of the nodes in community i in the network. According to the constraints given by [43], figure 11 just gives a simple example of the most modular network with $m = 5$, $L = 25$, $N = 20$ and $L_i = 4$ for all i. In this case, we get $Q = 0.6$ which means that this network holds quite strong community structure.

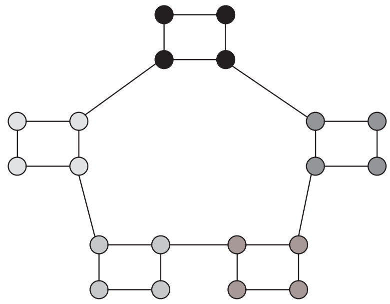

Figure 11. The most modular network with the total number of links $L = 25$. There are five communities (cliques), each of which is denoted by the circles with the same gray degree.

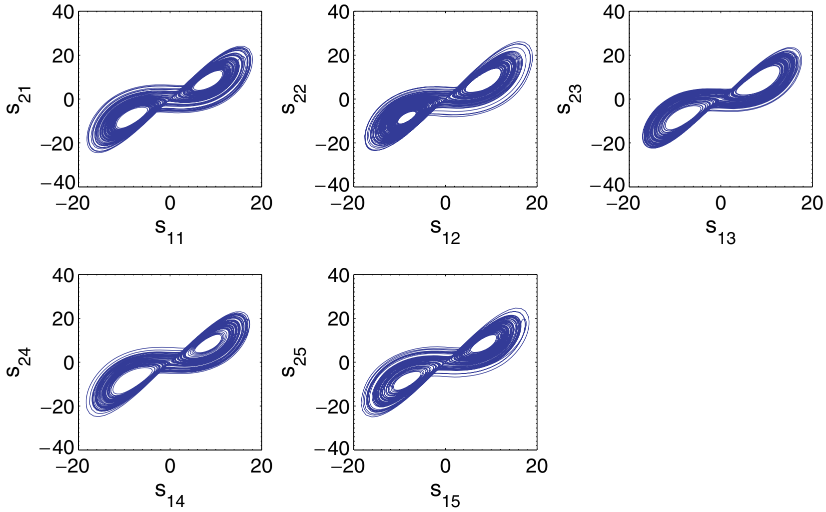

Figure 12. The desired five different states $s_i, i = 1, \ldots, 5$ for the controlled modular network. They are all with chaotic behaviors in their two-dimension phase space.

Next, we shall verify our pinning scheme with the control given by (10) under this modular network. The local dynamics of nodes, inner-coupling matrix and control gains are all set the same as the case in section 4.1. Under the partition in figure 11, there are five communities with the identical size $\sharp(U_i) = 4, i = 1, 2, \ldots, 5$, and $\sharp(\bigcup_{i=1}^{5} \widetilde{U}_i) = 10$. Then, these ten nodes in the set $\bigcup_{i=1}^{5} \widetilde{U}_i$ must be controlled according to the analysis in sections 2 and 3. The errors are defined as

$$E_1(t) = \sum_{i=1}^{m} \sum_{j \in U_i} \|x_j(t) - s_i(t)\|,$$

$$E_{ij}(t) = \|x_{i_0}(t) - x_{j_0}(t)\|, \qquad i_0 \in U_i, \quad j_0 \in U_j, \tag{31}$$

where the desired states $s_i, i = 1, 2, \ldots, 5$, are chosen as five different chaotic orbits of the Lorenz system (see figure 12), with initial conditions $(14 + 0.5i, 15 + 0.5i, 16 + 0.5i), i = 1, 2, \ldots, 5$, respectively. Figure 13 shows the simulation result with coupling strength $c = 71$ and control gain $d = 100$. By a similar analysis as done in section 4.1, we choose $\mathbf{P} = \mathbf{I}_3$ and

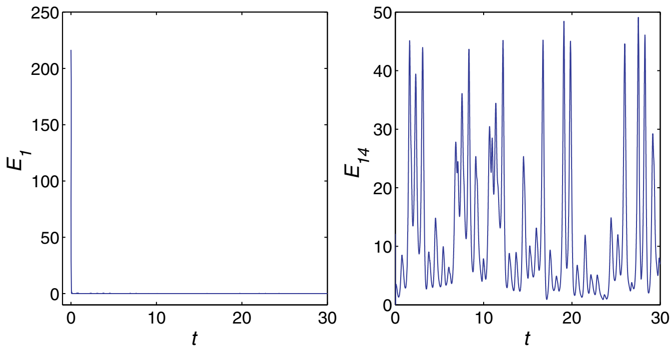

Figure 13. E_1 denotes the error of cluster synchronization of the controlled modular network, E_{ij} the error between U_i and U_j, corresponding to $c = 71$ and $d = 100$. Here, only the error E_{14} is shown for simplicity.

$\Delta = \delta\mathbf{I_3}$ with $\delta = 70.1$, and then get $c\lambda_1 + \delta = -0.19 < 0$. By theorem 1, the synchronization of these five clusters are globally stable.

Generally, it is hard to get analytical bounds for c and d to meet the conditions in theorem 1, especially for the unknown structure of the coupled system. Obviously, one can always realize the cluster synchronization globally if the coupling strength c and control gain d are chosen to be large enough, resulting in $c\lambda_1(d) \to -\infty$. However, this is rigorous for many practical applications. From the above simulation-based analysis in section 4.1, the cluster synchronization can be actually realized only by small coupling strength and control gain. This means that our pinning control scheme is very effective, but a more moderate condition should be found.

Other various kinds of cluster synchronization can also be realized by using the above pinning strategy. Certainly, it first depends on the corresponding partitions to the network. Moreover, in view of practical applications, we should guarantee that the controlled nodes are just a fraction of total nodes in network to reduce the cost for control. In conclusion, by these analytical and numerical simulations, the efficiency of the pinning control scheme provided in this paper has been verified.

5. Conclusions and remarks

In this paper, cluster synchronization of a class of controlled dynamical networks with community structure has been theoretically and numerically studied. The local stability and global stability of the cluster synchronization manifold have been investigated. The pinning control scheme provided in this paper mainly depends on the topology structure of communities of a given network. In order to achieve cluster synchronization, we should pin at least those nodes with direct connections between groups in a network with community structure. Since both the tree-shaped network and the most modular network possess well property of community structure, we take them as special examples to show how the pinning scheme influences the generation of clusters. The simulation results verify the efficiency of the pinning scheme introduced in this paper.

In order to achieve expected cluster synchronization, the pinning scheme provided in this paper is not unique. For example, in addition to the nodes in the set $\bigcup_{i=1}^{m} \widetilde{U}_i$ we may control additional nodes outside this set. With the increase of controlled nodes, it is easy to see that the minimal control gain $d_{\min}$ and critical value c^* for coupling strength would decrease. Generally speaking, small $d_{\min}$ and c^* are beneficial for the application of control techniques. However, the increase of the number of controlled nodes will also increase the cost of control. So, considering the global cost of control, the strategy introduced in this paper may be the most suitable. Other patterns of partial synchronization may also be realized by this kind of pinning control. For example, we only consider the complete synchronization of a part of nodes in a network and disregard the dynamics of the remaining nodes. In order to decrease the coupling strength and control gain, adaptive control may be a good choice. We will concentrate on these topics in future. In conclusion, these studies may promote the development of controlling chaos, communication engineering and other fields of science and technology.

Acknowledgments

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